All fractional Shapiro steps in the RSJ model with two Josephson harmonics

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RSJ model

Equations of the RSJ model [Aslamazov, Larkin, Ovchinnikov (1968); Stewart (1968); McCumber (1968)]:

$$\begin{cases} V = \dfrac{\hbar}{2e}\dfrac{\partial \varphi}{\partial t} \\ I = V/R + I_s(\varphi) \end{cases}$$

ac component of the current I: $I_{\mathrm{ac}} \cos(\omega t + \beta)$

Dimensionless units:

$$\tau = \omega t, \quad V_0 = \hbar\omega/2e, \quad I_0 = V_0/R$$
$$v = V/V_0, \quad j = I/I_0$$

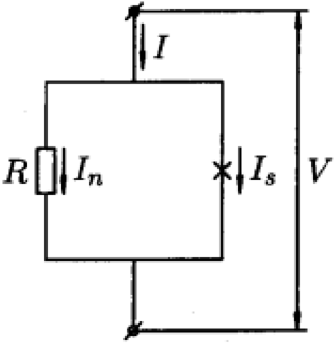


Figure: Scheme of the RSJ model.

Current-driven regime. One harmonic in the CPR

$$\dot{\varphi} + j_1 \sin\varphi = j_{dc} + j_{ac}\cos(\tau + \beta), \qquad \overline{v} = \overline{\dot{\varphi}}$$

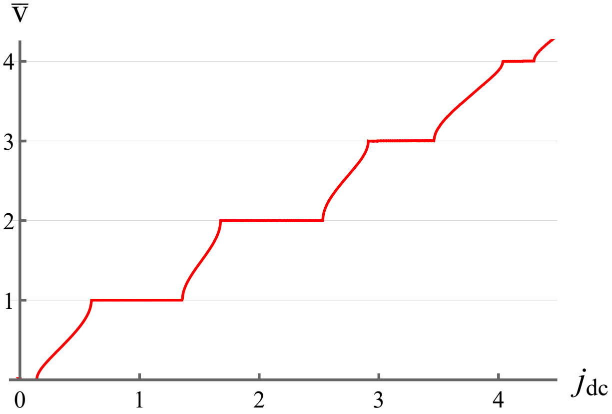


Figure: CVC at $j_{ac} = 3$. Only integer Shapiro steps are seen.

Integer Shapiro steps:

$$2e\overline{V} = n\hbar\omega$$

$$\overline{v} = n$$

Fractional steps
(are absent on the CVC):

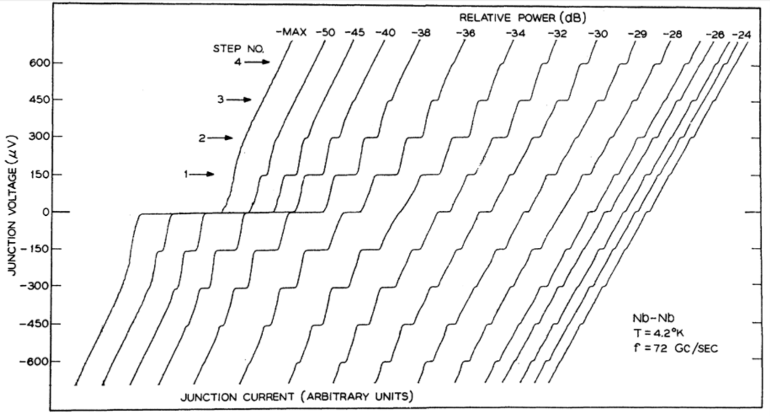
$$\overline{v} = n/k$$

Shapiro steps in experiment

PHYSICAL REVIEW VOLUME 169, NUMBER 2 10 MAY 1968

Millimeter-Wave Mixing with Josephson Junctions

C. C. Grimes and Sidney Shapiro*

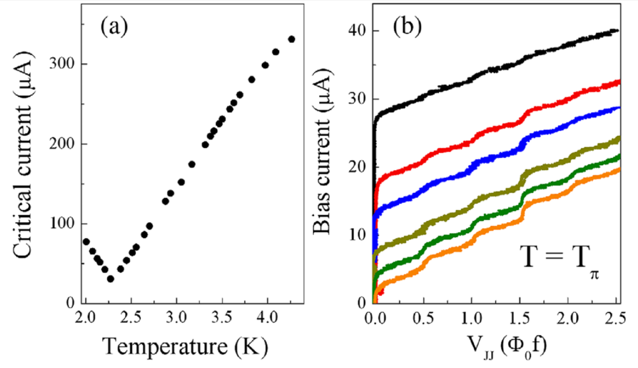


Half-integer Shapiro steps due to the second Josephson harmonic

PHYSICAL REVIEW LETTERS **121**, 177702 (2018)

Second-Harmonic Current-Phase Relation in Josephson Junctions with Ferromagnetic Barriers

M. J. A. Stoutimore,[1] A. N. Rossolenko,[2] V. V. Bolginov,[2,3,4] V. A. Oboznov,[2] A. Y. Rusanov,[2] D. S. Baranov,[2,5] N. Pugach,[3,6] S. M. Frolov,[7] V. V. Ryazanov,[2,4,8] and D. J. Van Harlingen[1]

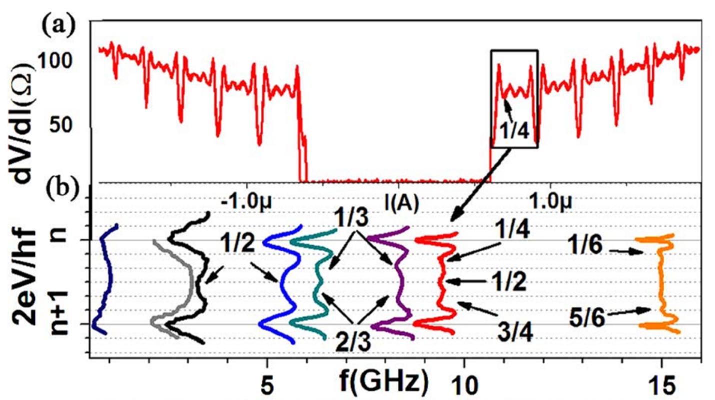


Higher-order Shapiro steps due to multiple Josephson harmonic

APPLIED PHYSICS LETTERS **93**, 192505 (2008)

Fractional order Shapiro steps in superconducting nanowires

R. C. Dinsmore III,[a)] Myung-Ho Bae, and A. Bezryadin



Voltage-driven regime

Fixed voltage $v(\tau) = v_{\mathrm{dc}} + v_{\mathrm{ac}} \cos\tau$ [Barone, Paterno — Textbook]:

$$\dot{\varphi} = v \quad \Rightarrow \quad \varphi(\tau) = \Delta\varphi + v_{\mathrm{dc}}\tau + v_{\mathrm{ac}} \sin\tau$$

CPR contains the second Josephson harmonic:

$$j_{\mathrm{s}}(\varphi) = j_1 \sin\varphi + j_2 \sin 2\varphi$$

Fourier series of j_s (J_n are Bessel functions):

$$j_s(\tau) = j_1 \sum_{n=-\infty}^{\infty} J_n(v_{\mathrm{ac}}) \sin\left((v_{\mathrm{dc}} + n)\tau + \Delta\varphi\right) + j_2 \sum_{n=-\infty}^{\infty} J_n(2v_{\mathrm{ac}}) \sin\left((2v_{\mathrm{dc}} + n)\tau + 2\Delta\varphi\right)$$

$$\bar{j} = v_{\mathrm{dc}} + \begin{cases} j_1 J_{-n}(v_{\mathrm{ac}}) \sin\Delta\varphi + j_2 J_{-2n}(2v_{\mathrm{ac}}) \sin 2\Delta\varphi, & \text{if } v_{\mathrm{dc}} = n \\ j_2 J_{-n}(2v_{\mathrm{ac}}) \sin 2\Delta\varphi, & \text{if } v_{\mathrm{dc}} = n/2 \\ 0 & \text{otherwise} \end{cases}$$

Trivial spikes (the fraction corresponds to the harmonic number): $v_{\mathrm{dc}} = n$ or $v_{\mathrm{dc}} = n/2$.
Presence of the kth Josephson harmonic leads to the fractional spikes of type n/k.

Current-driven regime. Two harmonics in the CPR

$$\dot{\varphi} + j_1 \sin\varphi + j_2 \sin 2\varphi = j_{\rm dc} + j_{\rm ac} \cos(\tau + \beta)$$

In the voltage-driven regime, there is a strict correspondence between the number of the Josephson harmonic and the fractional spikes. And what about the current-driven regime?

In addition to the Josephson effect, equation of the RSJ model arises in the following problems:

- charge transport by charge density waves in Peierls conductors [Artemenko, Volkov (1981); Wonneberger (1983); Grüner, Zettl (1985)]
- motion of mechanical systems such as an overdamped pendulum (mechanical analog of the JJ) [Sullivan, Zimmerman (1971)]
- Suslov system (specific type of a rigid-body motion) [Bizyaev, Mamaev (2020)]
- mathematically, the RSJ model is studied as an example of nonlinear dynamics [Buchstaber, Tertychniy (2013); Glutsyuk, Rybnikov (2016); Alexandrov, Glutsyuk, Gorsky arXiv:2504.20181]

$$\dot{\varphi} + j_1 \sin \varphi + j_2 \sin 2\varphi = j_{\rm dc} + j_{\rm ac} \cos(\tau + \beta)$$

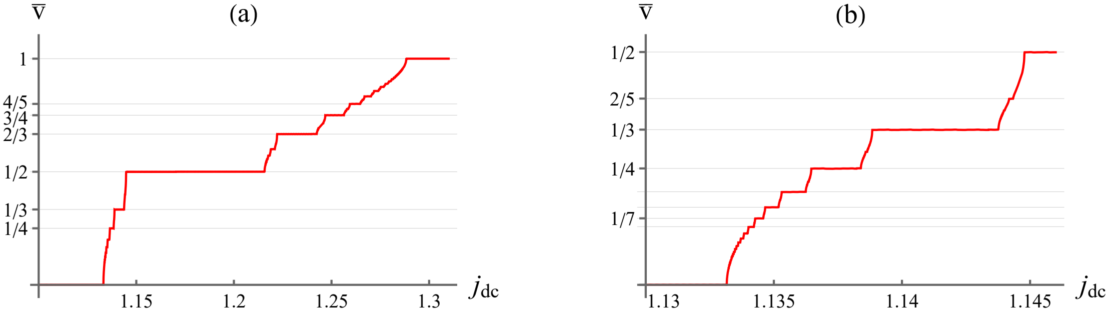


Figure: CVC at $j_{\rm ac} = 0.8$, $j_1 = 1$, and $j_2/j_1 = 0.7$. (a) Range of voltages below the first Shapiro step. (b) Range of voltages below the 1/2 Shapiro step [zoomed part of plot (a)].

Our purpose: to show that only two Josephson harmonics in the CPR are sufficient to produce all possible fractional Shapiro steps within the RSJ framework.

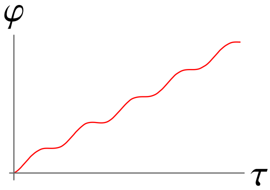
Perturbation theory with feedback

Perturbation theory with feedback with respect to small parameter $\alpha \ll 1$ [Thompson (1973)] (each limiting case has its own small parameters, for example, $j_{\mathrm{ac}} \ll 1$, $A = j_2/j_1 \ll 1$):

$$\varphi(\tau) = \varphi_0(\tau) + \varphi_1(\tau) + \varphi_2(\tau) + \dots,$$

$$j_{\mathrm{dc}} = j_{\mathrm{dc}}^{(0)} + j_{\mathrm{dc}}^{(1)} + j_{\mathrm{dc}}^{(2)} + \dots,$$

where φ_n, $j_{\mathrm{dc}}^{(n)} \sim \alpha^n$.



Only φ_0 contains linear growing part, $\overline{\dot{\varphi}_0} \neq 0$.
Then φ_n, $n \geq 1$ contain only oscillating terms, $\overline{\dot{\varphi}_n} = 0$.
To ensure this condition, we adjust the value of the current, introducing corrections $j_{\mathrm{dc}}^{(n)}$.

Thus, the corrections do not grow with time and always remain small.
As the result, we obtain the CVC as a parametric dependence $\overline{v}(j_{\mathrm{dc}})$ with parameter $j_{\mathrm{dc}}^{(0)}$:

$$\overline{v} = \overline{\dot{\varphi}} = \overline{\dot{\varphi}_0}$$

Limit of weak external irradiation and small second harmonic ($j_{\rm ac} \ll j_s$, $A \ll 1$)

$$\dot{\varphi} + j_1 \sin\varphi + j_2 \sin 2\varphi = j_{\rm dc} + j_{\rm ac} \cos(\tau + \beta), \qquad A = j_2/j_1$$

First step — perturbation theory with respect to $j_{\rm ac}/j_1 \ll 1$:

$$\varphi(\tau) = \varphi_0(\tau) + \underbrace{\varphi_1(\tau)}_{\sim j_{\rm ac}/j_1} + \underbrace{\varphi_2(\tau)}_{\sim (j_{\rm ac}/j_1)^2} + \ldots$$

$$j_{\rm dc} = j_{\rm dc}^{(0)} + j_{\rm dc}^{(1)} + j_{\rm dc}^{(2)} + \ldots$$

φ_0 is the solution of equation $\dot{\varphi}_0 + j_1 \sin\varphi_0 + j_2 \sin 2\varphi_0 = j_{\rm dc}^{(0)}$

Limit of weak external irradiation and small second harmonic ($j_{\mathrm{ac}} \ll j_s$, $A \ll 1$)

$$\dot\varphi + j_1 \sin\varphi + j_2 \sin 2\varphi = j_{\mathrm{dc}} + j_{\mathrm{ac}} \cos(\tau + \beta), \qquad A = j_2/j_1$$

First step — perturbation theory with respect to $j_{\mathrm{ac}}/j_1 \ll 1$:

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$$j_{\mathrm{dc}} = j_{\mathrm{dc}}^{(0)} + j_{\mathrm{dc}}^{(1)} + j_{\mathrm{dc}}^{(2)} + \dots$$

φ_0 is the solution of equation $\dot\varphi_0 + j_1 \sin\varphi_0 + j_2 \sin 2\varphi_0 = j_{\mathrm{dc}}^{(0)}$

Second step — perturbation theory with respect to $A \ll 1$:

$$\varphi_0(\tau) = f_0(\tau) + A f_1(\tau) + A^2 f_2(\tau) + \dots$$

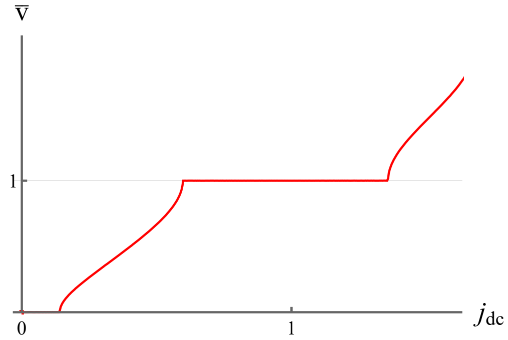
$$j_{\mathrm{dc}}^{(0)} = j_0 + A j_1' + A^2 j_2' + \dots$$

f_0 is the solution of equation $\dot f_0 + j_1 \sin f_0 = j_{\mathrm{dc}}^{(0)}$

$$\overline{\dot f_0} = \overline{v} = \nu \equiv \sqrt{j_0^2 - j_1^2}, \text{ where } \alpha = \arctan \nu/j_1$$

Integer step in the limiting case of small amplitude of the ac current and small amplitude of the second Josephson harmonic ($j_{ac} \ll j_s$, $A \ll 1$)

Current correction of the first order $j_{dc}^{(1)} = -\dfrac{\overline{j_{ac}\cos(\tau+\beta)\left(\frac{1}{\dot\varphi_0}\right)}}{\overline{1/\dot\varphi_0}}$



Single-harmonic case ($A = 0$, $\dfrac{1}{\dot\varphi_0} = \dfrac{1}{\dot f_0}$):

$$\frac{1}{\dot f_0} = \frac{j_0 + j_1\cos(\nu\tau - \alpha)}{\nu^2}$$

$$j_{dc}^{(1)} = -\frac{j_{ac}j_1}{2j_0}\cos(\beta+\alpha)\ , \text{ if } \nu = \pm 1$$

First integer Shapiro step [Thompson (1973)]:

$$\Delta j_{\pm 1} = j_{ac}j_1/j_0$$

Fractional steps in the limiting case of small amplitude of the ac current and small amplitude of the second Josephson harmonic ($j_{\rm ac} \ll j_s$, $A \ll 1$)

Current correction of the first order $j_{\rm dc}^{(1)} = -\dfrac{\overline{j_{\rm ac}\cos(\tau+\beta)(\frac{1}{\dot{f}_0} - \frac{A\dot{f}_1}{\dot{f}_0^2})}}{\overline{1/\dot{f}_0}}$

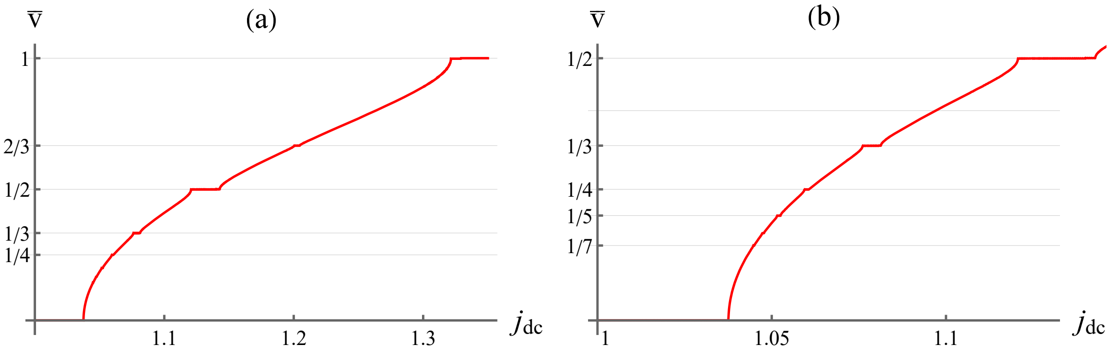
$$f_1(\tau) = \dot{f}_0 \int^{\tau} (j_1' - j_1 \sin 2f_0)\frac{dt}{\dot{f}_0} = \frac{\frac{2\nu^2}{j_1}\ln[j_0 + j_1\cos(\nu\tau - \alpha)] - 2j_0\cos(\nu\tau - \alpha)}{j_0 + j_1\cos(\nu\tau - \alpha)}$$

In the case $A \neq 0$, all possible harmonics appear $\frac{A\dot{f}_1}{\dot{f}_0^2} = \sum_k C_k e^{ik\nu\tau}$, and $j_{\rm dc}^{(1)} \neq 0$, if $\nu = \pm 1/k$

Amplitude of the steps [Ilyashenko, Ryzhov, Filimonov (2011)]:

$$\Delta j_{\pm 1/k} = 2\frac{Aj_{\rm ac}}{\sqrt{k^2 j_1^2 + 1}}\left(\frac{(\sqrt{k^2 j_1^2 + 1} - 1)^{k-1}}{(k-1)(kj_1)^{k-1}} - \frac{(\sqrt{k^2 j_1^2 + 1} - 1)^{k+1}}{(k+1)(kj_1)^{k+1}}\right)$$

Fractional steps in the limiting case of small amplitude of the ac current and small amplitude of the second Josephson harmonic ($j_{\mathrm{ac}} \ll j_s$, $A \ll 1$)



Large k $$\Delta j_{\pm 1/k} \simeq \frac{4 A j_{\mathrm{ac}} e^{-1/j_1} (j_1 + 1)}{k^3 j_1^2}$$

Higher orders of the perturbation theory in the case $j_{\mathrm{ac}} \ll j_s$, $A \ll 1$

$$\Delta j_{\pm n/k} \propto A j_{\mathrm{ac}}^n$$

- nth order with respect to j_{ac}
- higher orders of the perturbation theory with respect to A only produce small corrections to the result

Limit of large dc current ($j_{\rm dc} \gg j_s$)

$$\dot\varphi + j_1 \sin\varphi + j_2 \sin 2\varphi = j_{\rm dc} + j_{\rm ac}\cos\tau$$

Small parameters: $j_1/j_{\rm dc} \ll 1$, $j_2/j_{\rm dc} \ll 1$

Perturbation theory with respect to $j_s/j_{\rm dc} \ll 1$:

$$\varphi(\tau) = \varphi_0(\tau) + \underbrace{\varphi_1(\tau)}_{\sim j_s/j_{\rm dc}} + \underbrace{\varphi_2(\tau)}_{\sim (j_s/j_{\rm dc})^2} + \dots$$

$$j_{\rm dc} = j_{\rm dc}^{(0)} + j_{\rm dc}^{(1)} + j_{\rm dc}^{(2)} + \dots$$

Zeroth order: $\varphi_0 = j_{\rm dc}^{(0)}\tau + j_{\rm ac}\sin\tau + \Delta\varphi$ — similar to the voltage-driven regime!
The arbitrary phase shift between the junction dynamics and the external signal is now controlled by $\Delta\varphi$.

First order with respect to $j_s/j_{\rm dc}$ — only trivial Shapiro steps $\overline{v} = n$ and $\overline{v} = n/2$.
Higher orders — nontrivial steps.

Comparison of step amplitudes in the limiting case $j_s/j_{\mathrm{dc}} \ll 1$

Nontrivial steps due to the interplay of the two Josephson harmonics:

$$\Delta j_{\pm n/3} = 3 j_1 j_2 \left| \sum_{m=-\infty}^{\infty} \frac{J_m(j_{\mathrm{ac}}) J_{-n-m}(2 j_{\mathrm{ac}})}{n+3m} \right|$$

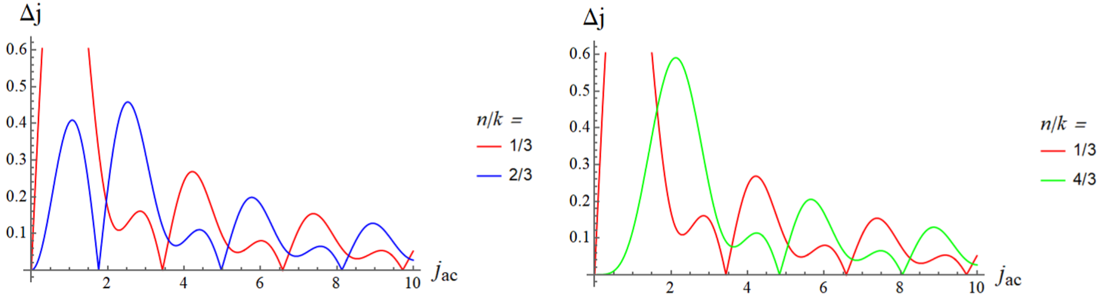


Figure: Comparison of steps amplitudes

Third order with respect to $j_s/j_{\rm dc} \ll 1$

$$\dot{\varphi}_3 = j_{\rm dc}^{(3)} - (j_1 \cos\varphi_0 + 2j_2 \cos 2\varphi_0)\varphi_2 + (j_1 \sin\varphi_0 + 4j_2 \sin 2\varphi_0)\varphi_1^2/2$$

$$\Delta j_{\pm n/4} = 8 j_1^2 j_2 \left| \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{J_m(j_{\rm ac}) J_l(j_{\rm ac}) J_{-n-m-l}(2j_{\rm ac})}{(n+4m)(n+4l)} \right|$$

$1 + 1 + 2 = 4$

$$\Delta j_{\pm n/5} = \frac{25}{4} j_1 j_2^2 \left| \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{J_m(2j_{\rm ac}) J_l(2j_{\rm ac}) J_{-n-m-l}(j_{\rm ac})}{(2n+5m)(2n+5l)} \right|$$

$1 + 2 + 2 = 5$

In higher orders of the perturbation theory:

$$\Delta j_{\pm n/2k} \propto j_1^2 j_2^{k-1}, \qquad \Delta j_{\pm n/(2k+1)} \propto j_1 j_2^k$$

Overlap of the two limiting cases

Conditions of overlapping: $j_{\mathrm{ac}}, j_2 \ll j_1 \ll 1$.

$$\Delta j_{\pm 1/k} \propto j_1^{k-2} j_2 j_{\mathrm{ac}} \qquad \text{vs} \qquad \begin{cases} \Delta j_{\pm 1/2k} \propto j_1^2 j_2^{k-1} j_{\mathrm{ac}}, \\ \Delta j_{\pm 1/(2k+1)} \propto j_1 j_2^k j_{\mathrm{ac}} \end{cases}$$

For $k = 2$, 3, and 4 formulas give the same result, but for $k \geq 5$ answers are different. Why?

Overlap of the two limiting cases

Conditions of overlapping: $j_{\rm ac}, j_2 \ll j_1 \ll 1$.

$$\Delta j_{\pm 1/k} \propto j_1^{k-2} j_2 j_{\rm ac} \qquad \text{vs} \qquad \begin{cases} \Delta j_{\pm 1/2k} \propto j_1^2 j_2^{k-1} j_{\rm ac}, \\ \Delta j_{\pm 1/(2k+1)} \propto j_1 j_2^k j_{\rm ac} \end{cases}$$

For $k = 2$, 3, and 4 formulas give the same result, but for $k \geq 5$ answers are different. Why?

Answer: in the general case $\Delta j_{\pm 1/k} \propto j_1^{\alpha} j_2^{\beta} j_{\rm ac}$, where $\alpha + 2\beta = k$, α and $\beta > 0$.

If $j_1^2 \gg j_2$, then $\beta = \min = 1$ — answer from the limiting case of $j_{\rm ac}/j_1$, $A \ll 1$.

If $j_1^2 \ll j_2$, then $\alpha = \min = 1$ or 2 — answer from the limiting case of $j_s/j_{\rm dc} \ll 1$.

Generally:

$$\Delta j_{\pm n/k} \propto j_1^{k-2} j_2 j_{\rm ac}^n, \quad \text{if } j_1^2 \gg j_2,$$

$$\Delta j_{\pm n/2k} \propto j_1^2 j_2^{k-1} j_{\rm ac}^n, \qquad \Delta j_{\pm n/(2k+1)} \propto j_1 j_2^k j_{\rm ac}^n, \quad \text{if } j_1^2 \ll j_2.$$

Josephson diode effect

$$\dot{\varphi} + j_1 \sin\varphi + j_2 \sin(2\varphi - \widetilde{\phi}) = j_{\rm dc} + j_{\rm ac}\cos(\tau + \beta)$$

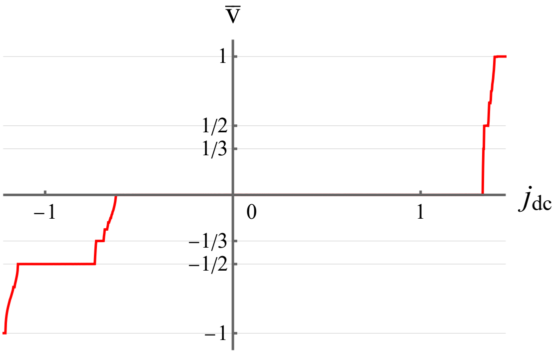


Figure: CVC at $j_{\rm ac} = 0.8$, $j_1 = 1$, $A = 0.7$, and $\widetilde{\phi} = \pi/2$. Positive and negative Shapiro steps are different.

Amplitude of the steps includes symmetric term (as if $\widetilde{\phi} = 0$) and asymmetric term, which arise in the next order of the perturbation theory.

Asymmetric corrections

In the limiting case $j_{\rm ac} \ll j_s$, $A \ll 1$:

$$\Delta j^{\rm asym}_{\pm 1/m} = \mp\gamma j_{\rm ac} A^2 \sin\widetilde{\phi}, \quad \gamma > 0.$$

In the limiting case $j_s \ll j_{\rm dc}$:

$$\Delta j^{\rm asym}_{\pm n/2k} = \pm(\gamma_1 j_1^4 j_2^{k-2} + \gamma_2 j_1^2 j_2^k)\sin\widetilde{\phi},$$

$$\Delta j^{\rm asym}_{\pm \frac{n}{2k+1}} = \pm(\gamma_3 j_1^3 j_2^{k-1} + \gamma_4 j_1 j_2^{k+1})\sin\widetilde{\phi}.$$

Conclusion: all fractional steps in the case of two-harmonic CPR

$$\dot{\varphi} + j_1 \sin\varphi + j_2 \sin 2\varphi = j_{\mathrm{dc}} + j_{\mathrm{ac}} \cos\tau$$

All fractional Shapiro steps $\overline{v} = \dfrac{n}{k}$

- Limit of weak external irradiation and small second Josephson harmonic ($j_{\mathrm{ac}} \ll j_s$, $j_2 \ll j_1$):

$$\Delta j_{\pm n/k} \propto A j_{\mathrm{ac}}^n$$

- Limit of large dc current ($j_s \ll j_{\mathrm{dc}}$):

$$\Delta j_{\pm n/2k} \propto j_1^2 j_2^{k-1}, \qquad \Delta j_{\pm n/(2k+1)} \propto j_1 j_2^k$$

- Josephson diode effect for fractional steps due to the phase shift between the two harmonics:

$$\Delta j^{\mathrm{asym}} \propto \sin\widetilde{\phi}$$

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